MODELING TASKS UNDER THE PERSPECTIVE OF 'GRUNDVORSTELLUNGEN'

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Abstract. In automated training in mathematics, as in the ASYMPTOTE project (Adaptive Synchronous Mathematics Learning Paths for Online Teaching in Europe), modelling tasks are a challenge in their creation and technical implementation. In modelling tasks, working with mathematics is concretised in the application area. Mathematical work is understood as a process of modelling: First, mathematical models are derived from a real problem; then the mathematical model is solved; finally, the mathematical solution is interpreted with regard to reality and the original problem is validated by the solution. This process focuses on the transition between the reality and the mathematical level. This paper focuses on this transition and its requirements and explains design principles of modelling tasks using examples from proportion and percentage calculation.

Key words: Grundvorstellungen, mental models, process of modeling.

MODELING AND APPLICATION

Starting with the assumption that competences can be learned, the question of different types of learning environments and the understanding of education at school is more interesting than mathematical precision. Klafki (1991) describes education as the ability of self-determination, where all-round education consists of three determining factors: (a) an education for all, (b) a general education and (c) an education with a focus to more general aspects. This aim of an all-round education, which was established by Winter (1976a, 1976b, 1996) in the area of mathematics conveys mathematics in three basic experiences. These experiences can be characterized as (E1) application orientation, (E2) structure orientation and (E3) problem orientation (cf. Blum & Henn, 2003; Winter, 1996). Thereby, 'application orientation' does not directly mean the preparation for specific situations in life, but rather, the possibility of a basic insight into nature, society and culture. 'Structure orientation' focuses more on the analysis of mathematical objects in relation to a deductive view of the world. The problem orientation on the other hand, emphasizes the acquisition of heuristic abilities to recognize and use samples in problem solving processes. However, these three aspects are connected with each other. Mathematics at school should provide the prerequisites to a basic mathematical knowledge in order to gain insights into various contexts of life in a reflective und understandable way. This view supports the ideas of Freudenthal (1973, 1981, 1983), whereby the arrangement of coherences is an essential aim of mathematical teaching at school. From these ideas, an understanding for mathematical competence as an individual characteristic has been developed according to mathematical literacy, which studies such as PISA are based on (OECD, 2022). This paper will focus on the requirements and design of modelling tasks, as these represent a separate task category in technical training exercises such as in the ASYMPTOTE project (Adaptive Synchronous Mathematics Learning Paths for Online Teaching in Europe), which must be implemented individually from a technical point of view (Oehler et al., 2023).

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The mathematical process of modeling

For modeling tasks, the view of mathematical literacy stands out due to an explicit application orientation. Grasping mathematical concepts should be taught by connecting it to real problems and how to use this acquired knowledge in real situations later in life (cf. Griesel, 1976). To solve a real-life mathematical problem, mathematical work can be understood as a process of modeling (cf. Blum, 1996). During this process, there can be separate different phases: (1) At first, the complexity of the real situation has to be focused to the specific problem in hand. You then get a model of reality. (2) This real model has to be transposed to a mathematical model on a mathematical level. (3) The mathematical model is solved and you get a mathematical result. (4) Finally the mathematical result is translated with a view to reality. Figure 1 shows those phases of a modeling process.

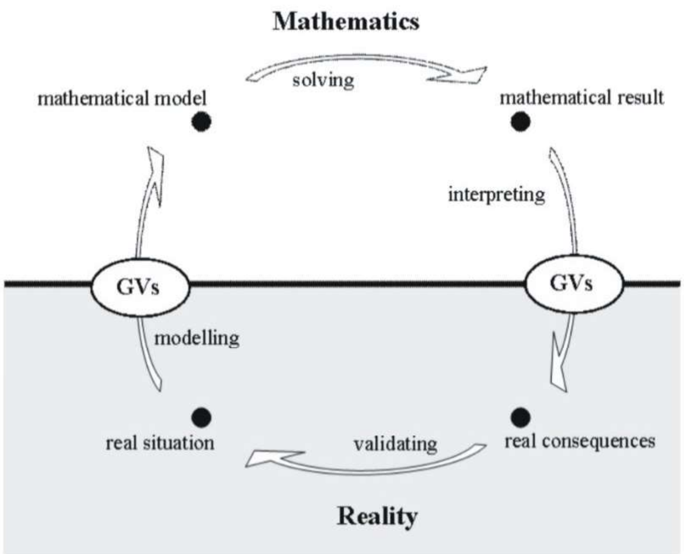

Figure 1: The process of modeling with the integrated concept of 'Grundvorstellungen' (GVs) (vom Hofe, Kleine, Blum & Pekrun 2005, p. 3)

The circulation indicates that the process should be learned by repetition due to the fact that mathematical models have to be modified or compared with other models. The correspondence of the perceived world with the mathematical model indicates the understanding of an individual in regard to nature, society and culture (cf. Winter & Haas, 1997). For further exposition of mathematical competences it is important to look more closely at the transition between reality and mathematics which play a decisive role in the description of mathematical literacy in the following section.

In this process of modeling, the transition between reality and mathematics represents a central task; a real situation is modeled, to the mathematical level and, the mathematical result is interpreted with respect to the real consequences. At first, we need mental objects of mathematical concepts for these transitions.

"I have avoided the term concept attainment intentionally. Instead I speak of the constitution of mental objects, which in my view precedes concept attainment and which can be highly effective even if it is not followed by concept attainment." (Freudenthal, 1983, p. 33)

The construction of such cognitive structures is described as the formation of 'Grundvorstellungen' (short "GV", e.g. vom Hofe, 1995, Greefrath et al., 2016, Salle & Clüver, 2021). This formation is indicated (a) by recording the meaning of new concepts about known structures, (b) by the construction of mental objects which represent the concept, and (c) by the application of new situations. The formation contains both the expansion and change of existing 'Grundvorstellungen' and the construction of new 'Grundvorstellungen' (cf. vom Hofe, 1995). 'Grundvorstellungen' therefore, describe fundamental mathematical concepts or methods and its interpretation into real situations. They describe the relations between mathematical structures, individual psychological processes and real situations. Kleine (2012) points out three features of this concept:

a) A clear relationship cannot exist between mathematical objects and specific 'Grundvorstellungen' because usually, mathematical objects are represented by several 'Grundvorstellungen' which are connected to each other.

b) One can distinguish 'Grundvorstellungen' in two ways: On the one hand, there exist primary 'Grundvorstellungen' which begin usually before mathematical instruction and these stand out due to concrete actions and concrete operations (e.g. the 'Grundvorstellungen' of the sum at a). On the other hand, secondary 'Grundvorstellungen' are developed during the time of mathematical instruction, which is indicated especially by mathematical representations (e.g. the 'Grundvorstellungen' of a function concept).

c) 'Grundvorstellungen' are neither fixed, nor used universally but are dynamic and are developing within a networked mental system. The necessity for the development results in a varying range of validity: If 'Grundvorstellungen' are sustainable in one mathematical area, they must be extended in another area. For example, 'Grundvorstellung' of multiplication when using different numbers for the second factor has separate results. With natural numbers the product is always higher than the first factor; with fractional numbers however, the product can be higher (2nd factor >1) or lower (2nd factor <1) than the first factor.

These characterizations point out, that 'Grundvorstellungen' can not be directly studied and require the need to be aware of the different types of behavior. This point of view marks the descriptive aspect of the concept: Through the analysis of individual behavior (e.g. at school, interviews, exams) the aim is to reconstruct the existing 'Grundvorstellungen' of mathematical objects. In contrast with this idea there is the normative aspect, whereupon 'Grundvorstellungen' are used as guidelines for the construction of mental objects of mathematical contents. The first aspect questions which 'Grundvorstellung' has been activated by a student; the second aspect questions which 'Grundvorstellung' has to be formatted by the student. If one compares the existing with the desired 'Grundvorstellungen', we have the idea outcome of an agreement. In many cases you can observe a deficit. This is a central topic of didactical research in the field of 'Grundvorstellungen'. Vom Hofe (2003) establishes the connection between basic ideas and basic education.

The previous explanations are intended to show that modelling tasks have a complex profile of requirements that must be taken into account when designing and evaluating tasks. It is therefore all the more important to structure modelling tasks clearly in order to be able to use this type of task adequately in a technical environment.

"GRUNDVORSTELLUNGEN' AS A THEORETICAL CRITERION FOR DESIGNING MODELING TASKS

In the first section, we took a deeper look on the transition between real situations and the mathematical level. Mental objects are necessary for those transitions as they mediate between reality and mathematics. Referring to this theoretical framework it is the aim of this section to deal with the question how 'Grundvorstellungen' can be used as a normative criterion for creating modeling tasks.

Firstly, we take a closer look at proportions, which are the mathematical concept of the following considerations. The importance of this mathematical content can be seen in the preparation of a simple mathematical model to mathematise many application-related situations. According to Kirsch (1969, 2002), proportions are only understood, if they are apprehended as a transformation which maintain the structure between two quantities. Thereby a transformation f between two quantities G1 and G2 is called proportional, if there is for any $n \in IN$ and any $a \in G1$: $f(n \cdot a) = n \cdot f(a)$.

In ideal cases, these kinds of transformations between reality and mathematics are based on two 'Grundvorstellungen' (cf. Malle, 2000):

- A connection between quantities can be described, founded or researched. Thereby, the elements of one quantity are in relation to elements of another quantity.
- The effect of the variation of elements of one quantity in relation to the elements of the other quantity is described or observed.

In Figure 2 we can see a simple example for a situation with a proportional context. The modelling structure of this item is the basis for the cognitive demand in these kinds of situations. It is due to this, that we can define the first cognitive level: (1) Relations between different quantities must be identified (on a "horizontal perspective"), i.e. the element of quantity G1 is in relation to the element of quantity G2. (2) The quantities are varying due to the characteristics of proportional transformations (on a "vertical perspective").

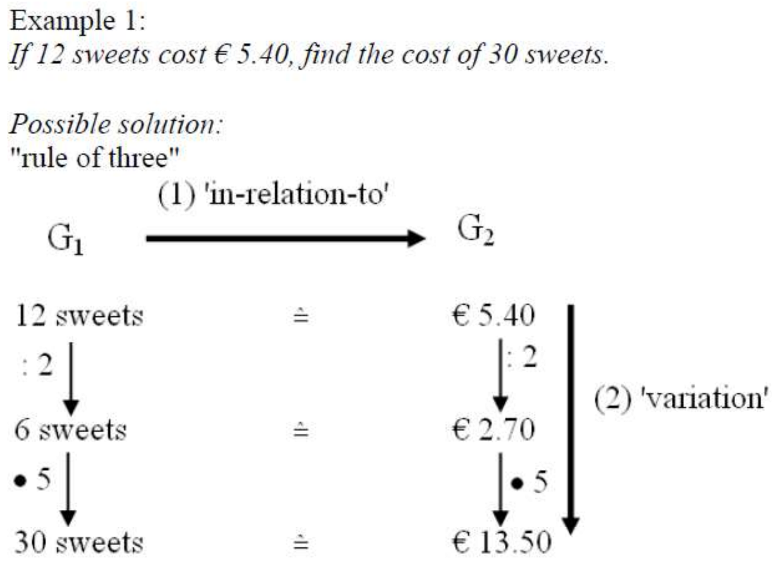

Figure 2: Identification of 'Grundvorstellungen' in an item on a basic level by the "rule of three" between two quantities G_1 and G_2. (Kleine, Jordan & Harvey, 2005, p. 234).

The 'Grundvorstellungen' can not only be integrated such as in Figure 2's solution; it is also possible to activate the 'Grundvorstellungen' for example by a factor of proportionality, by an equality of ratio or by an equality of quotient.

According to Griesel (1997), if we take a look at the demand in percentage calculations on this cognitive level, we can understand percentage quotations as special kinds of quantities. Percentages can be understood as a unit. Particularly with regard to percentage as a special case of fractions, we have obtained the following Grundvorstellungen' for percentage calculations (cf. Blum, vom Hofe, Jordan & Kleine, 2004):

- GV-Percentage 1. The whole is divided into one hundred equal-sized sections. For example, 43% of the students live in the town near to the school. By connecting the 'Grundvorstellungen' with an operation we talk about the hundredth-operator-GV.
- GV-Percentage 2. A statistical point of view whereby a basic set is divided into subsets with the cardinal number of 100. For example, 43% of the students live in the town near to the school means: for every 100 students of the school, 43 students live in the same town as the school.
- GV-Percentage 3. As mentioned above, percentages can be understood as a unit. Therefore, the unit and the value as a whole are a special kind of quantity.

Connected with these specific 'Grundvorstellungen' the basic items for percentage calculations can be solved similarly to the mentioned solution processes above. We can increase the requirement of items by adding further 'Grundvorstellungen' to the former demand in the previous level. This addition is not a trivial combination of 'Grundvorstellungen'. Typical items on this level are the combination of percentage calculations with arithmetical ones, such as 'addition' or 'subtraction' (cf. Padberg & Wartha, 2017). Figure 3 shows a possible solution for items using the process of the basic level above, whereby an additional structure of subtraction is needed.

An additional increase in requirement can be achieved by repeated combination of these aspects. For example, in compound interest calculations or the use of factors of growth in percentage calculations 'Grundvorstellungen' from the previous level have to be arranged in a non-trivial way. In practical work one can expect different kinds of concrete solution processes analogue to the former levels. However, from a theoretical point of view, we can describe the requirement as the interlinking of 'Grundvorstellungen'. The repetition in this level can be combined several times (e.g. within the compound interest calculation). This requirement can have an influence on the used solution method: Even with items which have multiple linking processes the use of operators is superior to other methods because of clarity and cognitive economy.

With regard to the design of modelling tasks as a technical training like in the ASYMPTOTE-project, the requirements arise that one takes into account these different steps of thinking, which can be seen in this example. These thinking steps have an influence on (1) the design of hints and (2) in the structuring of solutions as well as (3) in the structure of task sequences. Especially with regard to task batteries, it has proven helpful that tasks and the individual requirements are built up step by step so that the diagnosis of competences can be made unambiguously. A series of tasks in which example 2 follows tasks on simple percentage calculation and tasks like example 1 give clearer indications of pupils' abilities than example 2 alone would.

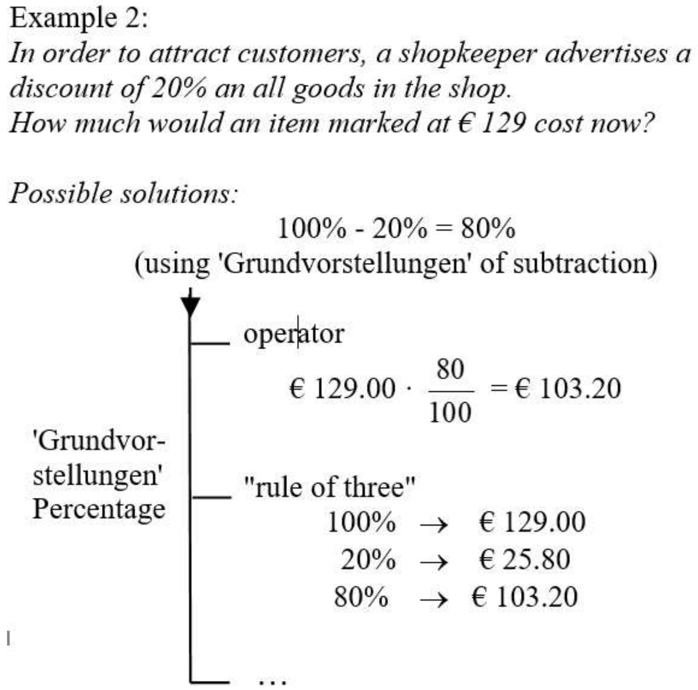

Figure 3: Identification of 'Grundvorstellungen' in an item with higher demand (Kleine, Jordan & Harvey, 2005, p. 236).

SUMMARY

This paper especially focussed on the transition between real situations and the mathematical level whilst working with modeling tasks. For these transitions mental objects are necessary which mediate between reality and mathematics. To be more precise we use the term 'Grundvorstellungen'. 'Grundvorstellungen' can be described as mental models for mathematical objects. We have shown that it is possible to use this concept as a theoretical criterion in order to describe items according to the extent of the 'Grundvorstellungen'. We have used the field of proportions and percentage calculations to illustrate our thoughts and ideas. We have developed in this field a (hierarchical) structure of three levels under the perspective of modeling tasks, whereby the demands on each level are determined by the extent of the used 'Grundvorstellungen'. For the design of modelling tasks, these requirements are an important guideline for the technical implementation. In order to implement competences and skills especially in a digital medium, one needs the form of didactic analysis presented here for the design of task series, solution instructions and sample solutions.

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