MODELING QUADRATIC FUNCTIONS IN THE SCHOOLYARD

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Abstract. *Teaching in the schoolyard and using digital technologies is one of a lot of components of the structure of mathematics lessons. One example of mathematical analysis lesson, which has been tested in class, will be presented in this paper: a basketball throw in correlation with quadratic functions. 15-year old pupils experience haptic mathematics, the functional graph arises using transparencies and pins, the functional equation can be developed and tasks can be solved in the fields of sports' application. Pupils use their own smartphones (bring your own mobile device) to this lesson: They film their activities and collect and evaluate the measured data of their video. Furthermore, this paper describes different ways of execution of this lesson and one possibility of interdisciplinary teaching approach in physics. The pupils compare the trajectory as a ballistic curve with a parabola of the quadratic function and they calculate the initial velocity.*

Key words: schoolyard, quadratic function, basketball throw, initial velocity

THE BASKETBALL THROW – ONE EXAMPLE OF MODELING MATHEMATIC

To create a math lesson which is creative, varied, exciting, close to reality, based on competences, differentiated and motivating for pupils is very difficult. A lot of pieces of the puzzle have to interlock. One of these puzzle pieces is learning in the schoolyard. Outdoor education may promote learning by improving pupils' attention, interest and enjoyment on acquisition of knowledge, and physical activity (Kuo & Jordan, 2019). If you have experienced contents, you don't have to learn them (Kramer, 2016).

Another puzzle piece is mathematics learning using mobile phones. The influence of the mobile phones on pupils' motivation is acknowledged in the literation (for example the study "Students' perceptions of Mathematics learning using mobile phones" of Baya'a & Daher). The benefits are learning mathematics in authentic real-life situations, visualizing mathematics and investigating it dynamically, performing diversified mathematical actions using new and advanced technologies and learning mathematics easily and efficiently (Baya'a & Daher, 2009).

In this described lesson, the mathematical modeling competence can be increased, a process of formulating real world situation in mathematical terms (Gablonsky & Lang, 2005). Mathematical modeling is a valuable and meaningful activity and it is important to introduce to mathematics learners in school (Ang, 2019). The pupils can understand that mathematics is useful and the basis for many professions, see the interdisciplinary nature of mathematics and understand what mathematical modeling means (Borromeo Ferri, 2018).

This lesson can be done if the pupils learned to graph quadratic equations in vertex form and to determine equation parameters. The materials for each group of three are one basketball, one smartphone, one pin, a part of transparency and one transparency pen.

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Stating the question

Does the trajectory of a basketball correspond to a parabola?

Figure 1: Analysis of a basketball throw as a parabolic trajectory.

This question was explored with a class of 15-year old pupils in Germany. The following elements were included in this lesson:

The first part of the lesson takes place in the schoolyard. Two pupils (first pupil and second pupil, see Figure 2) throw the ball to each other and a third pupil films this with his or her own phone. The pupil who films the action (third pupil, see Figure 2) should stand at the same distance from the other group members in order to film an undistorted trajectory. This pupil should also stand at a certain distance from the thrower and the catcher so that the central perspective approximately conforms to the parallel-perspective.

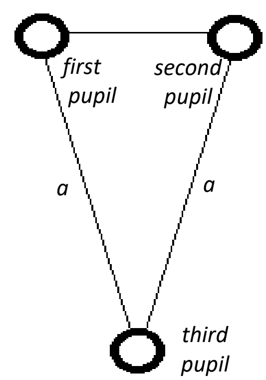


Figure 2: Position of the three pupils.

It should be noted that the ball is tossed from left to right if the footage is used for further researches. The pupils ought to throw a big ball (for example a basketball or a medicine ball), so that the trajectory can be seen clearly in the video afterwards. The ball should also be heavy so that the air friction has a low impact on the movement. It is possible that the pupils bring their own balls to school.

Curve fitting with quadratic function

After the basketball throws in the schoolyard, the class returns to the classroom and the pupils meet in their groups. It is important to discuss the perfect position of the coordinate system. Pupils can express their ideas and explain them mathematically. Effective communication is critical for more precise instruction and deeper mathematic learning and creating classroom environments in which pupils practice multiple forms of communication is imperative (Sammons, 2018).

For teaching more mathematical or physical issues, the teacher and the pupils agree on the origin of coordinates, which conforms the feet of the thrower. The body size (head and feet) is marked on the transparency to illustrate the scale.

Now the question arises how to draw the trajectory parabola as a graph of a function onto paper. The pupils take a transparency and place it on the screen of their phone. The learners draw the flight curve onto the transparency by playing the video in slow motion. Multiple loop and repeatedly stopping are another possibility. The pupils mark the points with little circles because the coordinates are approximate values.

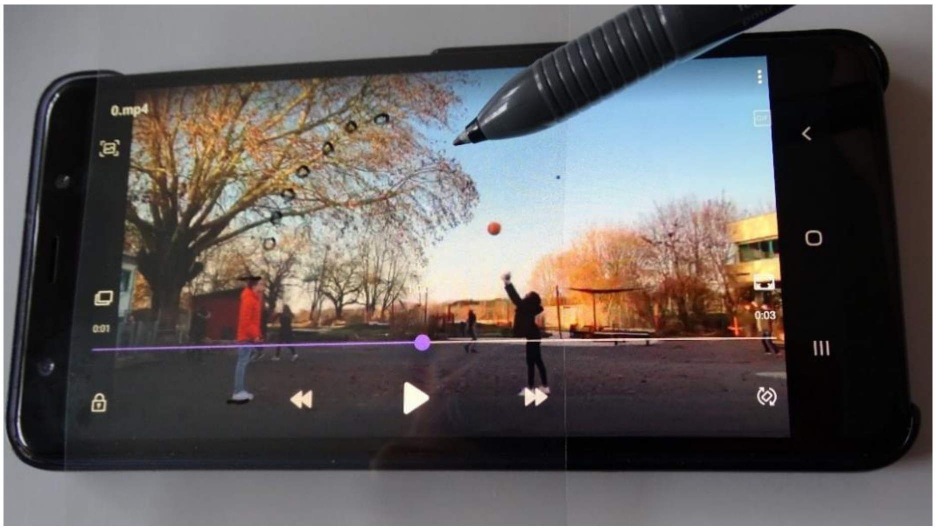


Figure 3: The points are marked on a transparency put onto the screen.

In a next step, the circles and the body size are transferred to millimeter paper, either with pins or by projection.

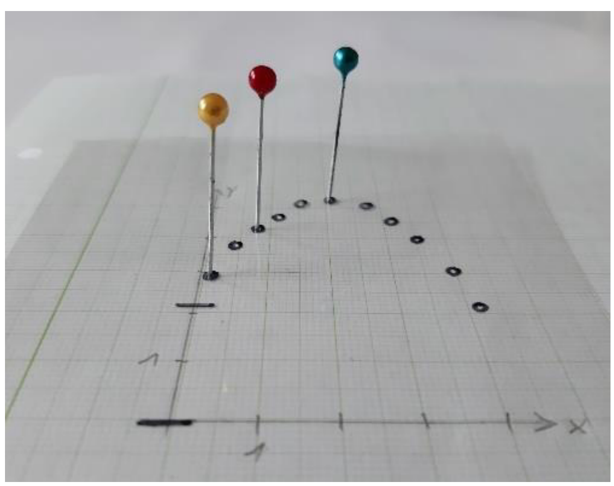


Figure 4: Pins are pierced through the transparency.

Alternatively, the pupils work with a document camera. They put their transparency onto the document camera, hold their mathematics exercise books against the wall, draw the projection of the trajectory in their exercise books and enlarge so their drawing to a corresponding size.

The pupils estimate the coordinates of the point of throw-out and the highest point as vertex. They substitute the points into the vertex form and reconstruct the quadratic function. Now, the learners calculate points of the function and draw the parabola in the same coordinate system.

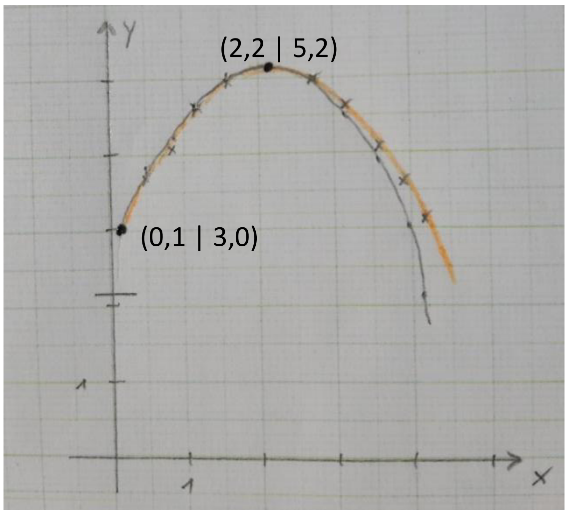


Figure 5: Trajectory (black) and parabola (orange)

Coordinate axes in unit of length.

A pupil's solution; see Figure 5:

$$f(x) = a \cdot (x - d)^2 + e$$

$$3{,}0 = a \cdot (0{,}1 - d)^2 + e$$

$$3{,}0 = a \cdot (0{,}1 - 2{,}2)^2 + 5{,}2$$

$$f(x) = -\frac{220}{441}(x - 2{,}2)^2 + 5{,}2$$

Interpret solution

The pupils detect the deviations from these two graphs. If the ball soars, the trajectory (the ballistic curve) is almost identical in the parabola (reconstructed with only two points). If the ball drops, the trajectory is below the parabola and the difference is greater, the longer the ball flies. The deviations may be pronounced in environmental influences, such as the friction between the air and the ball.

The learners can calculate the maximum height of the ball. The thrower tells his body size (as marked onto the transparency) and the proportion is determined between the body size and one unit of length of the squared paper or millimeter paper. Then they use the value of the y-coordinate of the vertex and, together with the proportion, calculate the maximum height.

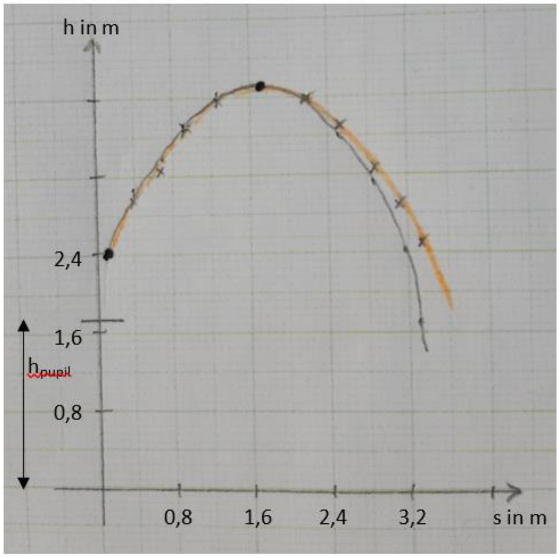


Figure 6: Drawing of the calculation of the maximum height of the ball (h_{pupil} is the body size of the thrower).

A pupil's solution; see Figure 6:

The thrower said that his body size is 1,76 m.

$$h_{pupil} = 1{,}76\ m \triangleq 2{,}2\ unit\ of\ length$$

$$one\ unit\ of\ length \triangleq 0{,}8\ m$$

$$vertex's\ y - coordinate:$$

$$5{,}2\ unit\ of\ length \triangleq 4{,}16\ m$$

The maximum height of the ball is approximately 4,2 m.

Another possibility to investigation the proportion is the distance between the thrower and the catcher. Both pupils mark their location with chalk in the schoolyard, measure the length and calculate the proportion between the distance and one unit of length of the squared paper or millimeter paper.

Another calculation is the study of the initial velocity v_0. The time of the throw is noticeable in the film. The pupils draw a tangent at the point of throw-out.

A pupil's solution; see Figure 7*:*

$$s_x = 3{,}2\ m\ \ (determined\ by\ the\ drawing)$$

$$t = 0{,}922\ s\ (determined\ by\ the\ video)$$

The velocity in x-direction v_x is constant:

$$v_x = \frac{s_x}{t} = \frac{3{,}2m}{0.922s} = 3{,}47\frac{m}{s}$$

Investigation v_0: The pupils draw with the tangent a rectangular triangle and measure the lengths of the arrows for v_0 and v_x.

$$\frac{v_0}{v_x} = \frac{5{,}5\ cm}{2{,}5\ cm} \qquad v_0 = \frac{5{,}5\ cm}{2{,}5\ cm} \cdot v_x$$

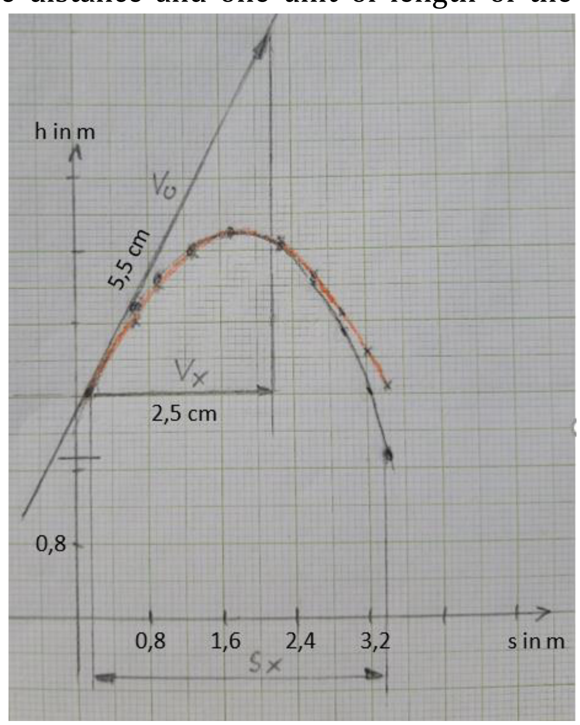


Figure 7: Drawing of the calculation initial velocity.

$$v_0 = \frac{5{,}5\ cm}{2{,}5\ cm} \cdot 3{,}47\frac{m}{s} = 7{,}63\frac{m}{s}\quad The\ initial\ velocity\ of\ the\ ball\ is\ approximately\ 7{,}63\frac{m}{s} \approx\ 27{,}5\ \frac{km}{h}$$

Second possibility to calculation the study of the initial velocity v_0: The pupils measure the angle between the arrows for v_0 and v_x. This part requires knowledge of trigonometric knowledge.

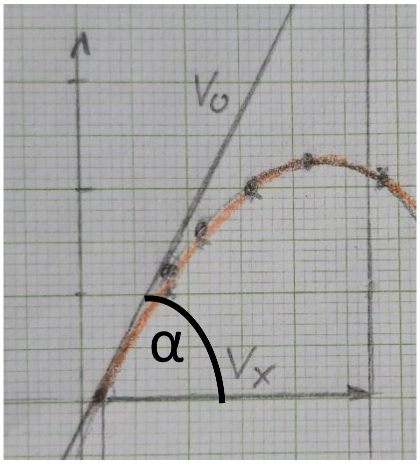


Figure 8: A part of Figure 7.

A pupil's solution; see Figure 8:

$$\alpha = 63°$$

$$\cos(\alpha) = \frac{v_x}{v_0}$$

$$v_0 = \frac{3{,}47\,\frac{m}{s}}{\cos(63°)}$$

$$= 7{,}64\,\frac{m}{s}$$

Third possibility for older pupils in use the first derivative:

A pupil's solution:

$$f(x) = -\frac{220}{441}(x-2{,}2)^2 + 5{,}2 \qquad f'(x) = -\frac{440}{441}(x-2{,}2) \qquad \tan(\beta) = f'(x_0)$$

See; Figure 5: $x_0 = 0{,}1 \quad f'(0{,}1) \approx 2{,}095 \quad \beta \approx 64{,}48° \quad v_0 = \frac{3{,}47\,\frac{m}{s}}{\cos(64{,}48°)} = 8{,}05\,\frac{m}{s}$

The quality of the results can be checked with video players for sports analysis. The program determines a position of the ball every one or two milliseconds.

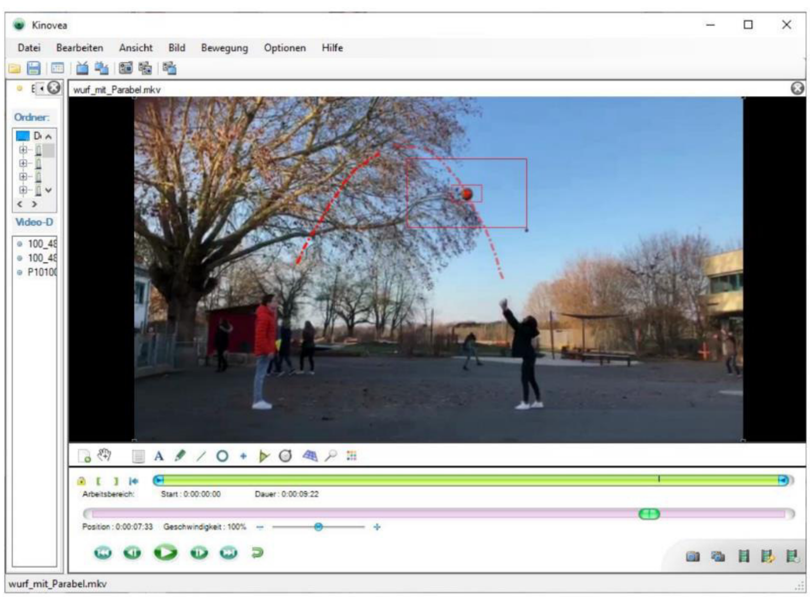
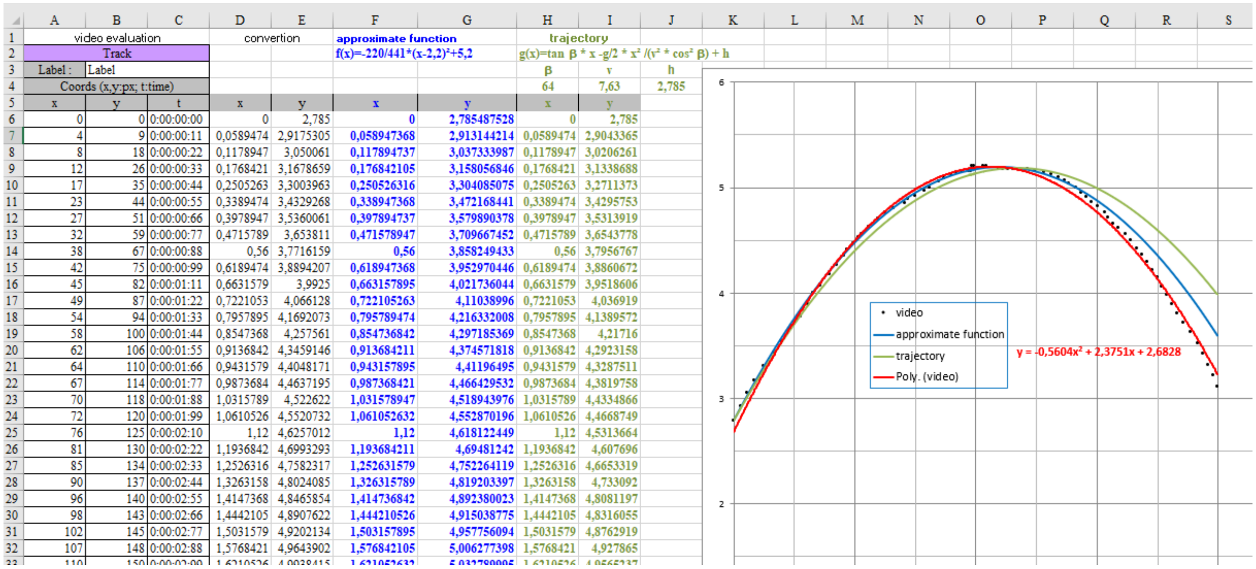


Figure 9: An example of a video player.

This data can be transferred to a spreadsheet program.



	A	B	C	D	E	F	G	H	I	J
1	video evaluation			convertion		approximate function		trajectory		
2	Track					f(x)=-220/441*(x-2,2)²+5,2		g(x)=tan β * x -g/2 * x² /(v² * cos² β) + h		
3	Label :	Label						β	v	h
4	Coords (x,y:px; t:time)							64	7,63	2,785
5	x	y	t	x	y	x	y	x	y	
6	0	0	0:00:00:00	0	2,785	0	2,785487528	0	2,785	
7	4	9	0:00:00:11	0,0589474	2,9175305	0,058947368	2,913144214	0,0589474	2,9043365	
8	8	18	0:00:00:22	0,1178947	3,050061	0,117894737	3,037333987	0,1178947	3,0206261	
9	12	26	0:00:00:33	0,1768421	3,1678659	0,176842105	3,158056846	0,1768421	3,1338688	
10	17	35	0:00:00:44	0,2505263	3,3003963	0,250526316	3,304085075	0,2505263	3,2711373	
11	23	44	0:00:00:55	0,3389474	3,4329268	0,338947368	3,472168441	0,3389474	3,4295753	
12	27	51	0:00:00:66	0,3978947	3,5360061	0,397894737	3,579890378	0,3978947	3,5313919	
13	32	59	0:00:00:77	0,4715789	3,653811	0,471578947	3,709667452	0,4715789	3,6543778	
14	38	67	0:00:00:88	0,56	3,7716159	0,56	3,858249433	0,56	3,7956767	
15	42	75	0:00:00:99	0,6189474	3,8894207	0,618947368	3,952970446	0,6189474	3,8860672	
16	45	82	0:00:01:11	0,6631579	3,9925	0,663157895	4,021736044	0,6631579	3,9518606	
17	49	87	0:00:01:22	0,7221053	4,066128	0,722105263	4,11038996	0,7221053	4,036919	
18	54	94	0:00:01:33	0,7957895	4,1692073	0,795789474	4,216332008	0,7957895	4,1389572	
19	58	100	0:00:01:44	0,8547368	4,257561	0,854736842	4,297185369	0,8547368	4,21716	
20	62	106	0:00:01:55	0,9136842	4,3459146	0,913684211	4,374571818	0,9136842	4,2923158	
21	64	110	0:00:01:66	0,9431579	4,4048171	0,943157895	4,41196495	0,9431579	4,3287511	
22	67	114	0:00:01:77	0,9873684	4,4637195	0,987368421	4,466429532	0,9873684	4,3819758	
23	70	118	0:00:01:88	1,0315789	4,522622	1,031578947	4,518943976	1,0315789	4,4334866	
24	72	120	0:00:01:99	1,0610526	4,5520732	1,061052632	4,552870196	1,0610526	4,4668749	
25	76	125	0:00:02:10	1,12	4,6257012	1,12	4,618122449	1,12	4,5313664	
26	81	130	0:00:02:22	1,1936842	4,6993293	1,193684211	4,69481242	1,1936842	4,607696	
27	85	134	0:00:02:33	1,2526316	4,7582317	1,252631579	4,752264119	1,2526316	4,6653319	
28	90	137	0:00:02:44	1,3263158	4,8024085	1,326315789	4,819203397	1,3263158	4,733092	
29	96	140	0:00:02:55	1,4147368	4,8465854	1,414736842	4,892380023	1,4147368	4,8081197	
30	98	143	0:00:02:66	1,4442105	4,8907622	1,444210526	4,915038775	1,4442105	4,8316055	
31	102	145	0:00:02:77	1,5031579	4,9202134	1,503157895	4,957756094	1,5031579	4,8762919	
32	107	148	0:00:02:88	1,5768421	4,9643902	1,576842105	5,006277398	1,5768421	4,927865	

Figure 10: Comparison of video data and calculated data.

Video data are in columns A, B and C. The x and y values of the video data are transformed using the maximum y value and the x value of the time of catching in columns D and E (see Figure 10 black points). The approximation of the values can be shown with a trend line. (see Figure 10 red curve). The x and y values of the approximate function f (see Figure 5) are in the columns F and G (see Figure 10 blue curve).

The explicit equation of the trajectory in the space area considering the height h (h = f(0)) is $g(x) = -\frac{g}{2} \cdot \frac{x^2}{v_0^2 \cdot cos^2\beta} + x \cdot tan\beta + h$. The x and y values of g are in columns H and I (Figure 10 green curve).

Results

The lesson works well if the pupils divide themselves into groups of three. All pupils enjoyed the work with their own smartphone and most pupils achieved good flight curves and good quadratic function approximations.

One group used a handball because a girl from this group plays handball in her free time and she brought her own ball to school. The handball was difficult to see in the video.

In another group, the trajectory was flat and this group had problems to reconstruct the quadratic function. It is important for the teacher to announce that the pass must be thrown up high into the air.

The evaluation with the computer shows that the different types of calculation, approximation of measured values every one or two milliseconds, reconstruct the quadratic function in use the point of throw-out plus the vertex and equation of the trajectory in use the angle and the initial velocity, lead to a similarly good result.

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