LEARNING MATH OUTDOORS: GRAPH THEORY USING MAPS

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Abstract. *In this paper, we present planning for a learning trajectory to teach a graph theory rule, specifically that underlying Euler's Königsberg Bridges problem, using a design research approach. With a Realistic Mathematics Education approach and contextualizing the learning trajectory in the embodied cognition theoretical framework, we designed and tested some tasks to be performed outdoors and with the use of the students' body and movement. Preliminary results show how students connected the outdoor activities with other tasks that required a higher level of abstraction, understanding some mathematical properties, referring to the experience they lived in first person. These results motivate further investigations on this topic.*

Embodied; RME; discrete mathematics; Graph theory; Eulerian paths; maps; navigation.

INTRODUCTION

Mathematics and Computer Science (CS) educators have been analyzing the possibility of using graph theory in a lower grade school setting for quite some time now. In some educational systems, it is considered almost a part of the traditional curriculum (e.g. Niman, 1975), while in others it begins to appear with some single activities proposed to the students as "extracurricular" (Ferrarello & Mammana, 2018). However, some factors as the distance from a traditional school curriculum, a lack of teacher training on the subject and a general uncertainty by teachers about trying new topics in a mathematics area they are not familiar with (Gaio & Di Paola, 2018), has not made it possible for these discrete mathematics topics to enter the school world with a shared acknowledgement yet. On the other level, the abstraction capability, with abstract thought as opposed to mental patterns, is considered in the literature proper to children only over 11 or 12 years of age (Piaget, as in Lister, 2011). Can any task stimulate the ability to abstract, at least partial, already in primary school age?

Purpose for this paper is to describe the implementation of a learning trajectory (as in Clements & Sarama, 2004) about a graph theory topic, specifically Eulerian paths, involving activities and tasks to be performed outdoor and in person, with the use of maps and navigation skills, combined with some paper and pencil tasks. The design of the learning trajectory follows a design research approach (Plomp & Nieveen, 2007) and the principles of Realistic Mathematics Education (RME, Gravemeijer, 1994) instruction theory. In this prospect, outdoor education in a suitable context gives a strength to RME, as living the situation in person makes the activity much more related to a valuable experience.

We will describe the theoretical framework and theories used as a background, then proceed with a description of the learning trajectory implemented and draw some conclusions from the data collected. The goal is to show how such an outdoor didactical approach can lead to a better understanding of the problem, a more realistic view of the mathematics involved and a deeper discovery of the mathematical rule by the students, grounded in their physical experience of the real world with their bodies.

Gaio, A., Branchetti, L. & Capone, R. (2020). Learning Math Outdoors: Graph Theory using Maps. In M. Ludwig, S. Jablonski, A. Caldeira, & A. Moura (Eds.), *Research on Outdoor STEM Education in the digiTal Age. Proceedings of the ROSETA Online Conference in June 2020* (pp. 95-102). Münster: WTM. https://doi.org/10.37626/GA9783959871440.0.12

Literature review

As mentioned, graph theory and discrete mathematics in general, has not to be so popular in the school world, especially at lower school grades (Gaio & Di Paola, 2018). Therefore, the need to produce a teaching-learning proposal which can be innovative and ease the approach to this area of mathematics. Even though we think it can be really interesting to present students with this kind of mathematics, which is a strong branch of mathematics and CS in modern research, the main reason for choosing the topic lies in the fact that it really can provide suitable math problem (e.g. CS Unplugged project, Bell et al., 2009) to enhance general problem solving abilities and real world mathematics. Also, activities in discrete mathematics "allow a kind of new beginning for students and teachers" (Goldin, 2004). Nonroutine problems are used to elicit mathematical reasoning processes and raise interest in both teachers and students, with new opportunities for mathematical discovery (ibid, 2004).

The replacement of algorithmic models with an emphasizing of the students' own construction of knowledge (Jonsson et al., 2014) is stressed and, also, students will be engaged in activities for which "they have to 'struggle' (in a productive meaning of the term) with important mathematics" (ibid., 2014). It is in this sense suggested that children's creative development and a wide range of skills can be learned in an authentic context such as the outdoors (Beard & Wilson, 2006).

Finally, curriculum recommendations and guidelines support this approach; see for example the European Union recently published recommendations (EU Council, 2018) where mathematical competence is "the ability to develop and apply mathematical thinking and insight in order to solve a range of problems in everyday situations".

THEORETICAL FRAMEWORK

As a theoretical framework for our research we choose that of embodied cognition (Lakoff & Núñez, 2000), since it states that the comprehension of abstract mathematical concepts is rooted in sensory-motor experiences and in interaction with the environment and the world (ibid., 2000). The conceptualization of abstract concepts, using "ideas and modes of reasoning grounded in the sensory-motor system" is called *conceptual metaphor* (ibid., 2000). Mathematical activity which is embodied, can result in a better understanding of abstract concepts (De Freitas & Sinclair, 2014) and also, cognition can be strongly linked to and actually "based in perception and action, and it is grounded in the physical environment" (Alibali & Nathan, 2012).

Such views can be related to some key aspects of the Realistic Mathematics Education (RME, Gravemeijer, 1994) instruction theory. The development of RME was all started by a Hans Freudenthal idea, that mathematics should be considered as a human activity (1991). From this significant idea, the principles of the RME theory was then formalized: guided reinvention, didactical phenomenology, and emergent models.

RME is an instructional approach that aims at bridging a gap between abstract mathematical concepts and the real world; mathematics is seen as a human activity and is therefore seen as connected to reality. Students are the actors of their learning and, guided by teachers and educators, enhance a process of reinvention of the mathematical process (ibid., 1994).

METHODOLOGY

For the design of the learning trajectory, the design research framework has been used (Plomp & Nieveen, 2007). Educational design research is the systematic study of designing, developing and evaluating educational interventions (such as programs, teaching-learning strategies and materials, products and systems) as solutions for complex problems in educational practice, which also aims at advancing our knowledge about the characteristics of these interventions and the processes of designing and developing them (Plomp & Nieveen, 2007).

The subject of discrete mathematics and graph theory and our aim seems to be a suitable environment to use such an approach. According to Clements and Sarama (2004), learning trajectories are constituted by three parts: (i) a specific mathematical goal, (ii) a path along which the children develop to reach that goal and (iii) a set of instructional activities that help move along that path. This structure looks simple yet complete to give educators a good understanding of both the math and the children concept development.

In our learning sequence, the specific goal (i) is to find out a rule to define if a given graph has or not a Eulerian path. A Eulerian path is a trail in a finite graph that visits every vertex exactly once. Apart from this, a mathematical goal we have in mind in the research is also a general improvement of collaborative problem solving and argumentative skills in the students.

The path (ii) to reach the goal is described as the developmental progression of thinking and learning in the children's understanding of the problem and of the general rule. We move from an informal understanding of the problem, to some tentative generalization of an odd/even rule, to the final formulation of a mathematical rule, appropriate for the age range of the students.

The set of instructional activities (iii) is described in the following paragraphs, presenting the sequence of tasks we used, after a double process of refinement in cycles of instructions, as in the principles of design research, going from a preliminary teaching experiment to an hypothetical learning trajectory and finally to the learning trajectory which follows.

Tasks of the learning trajectory

The preliminary teaching experiment and the learning trajectories here described were implemented in 4th, 5th and 6th grade classes in Italy, involving a total of 19 classes and around 380 students, during the school years 2016/2017, 2017/2018 and 2018/2019. The final version of the learning trajectory, which is presented, has been implemented in 8 of these, five 6th grade and three 5th grade classes, with a total of around 175 students involved in the research.

The tasks proposed are about Eulerian paths in a graph, connected with the well-known Königsberg bridges problem. The town is crossed by the Pregel river, with two islands in the middle. This makes Königsberg divided on two islands and two sides, interconnected by seven bridges. Is it possible to find a path crossing all seven bridges exactly once? The problem is impossible, but a proof for it lies in a property of certain graphs to have an even degree for every vertex (i.e. an even number of edges attached to it), except for at most two vertices which are the starting and the ending one of the paths. Examples of Eulerian paths tasks can also be drawing problems such as: "can you draw a given graph, without taking the pen off the paper, and without repeating the same vertex more than once?".

In our first teaching experiments, this kind of activity was implemented, trying to guide the students from particular cases towards a generalization of the "even degree rule", our (i) specific mathematical goal. The paper and pencil activities worked with some of the students, but resulted, in a majority of cases to be more abstract and not adherent to a problem student could refer to. "Why can we not jump from one vertex to another and do we have to stick the pen on the paper?" or "Why can't we pretend to cross the same bridge twice?" are some examples of reactions to the proposals. Therefrom, the need to try a different way and change the context and way of execution of these tasks. The connection was made with the possibility of moving in the space, recreating graphs in the city streets. We used a map for orienteering, a navigation outdoor sport requiring to find the best way between different control points, usually in a given order, interpreting the map and finding the right locations and routes. The use of such a detailed map allows students to work on particular topographical skills as map symbols and on their navigation ability.

We finally designed the tasks in the learning trajectory, mixing the outdoor and the paper and pencil tasks, to consider both the informal approach of the outdoor activities and the possibilities of later formalizing the concept in the more abstract paper requests.

Task (1) is used as a teaser for the next activities. Namely, the Königsberg bridges problem, as described above, is presented to students, who are let free to try to find a solution, which they obviously cannot. No big discussion is done on why it does not work, but the next tasks will guide us towards the solution.

Figure 1: Task (1), the Königsberg bridges problem.

Task (2) is proposed in the classroom as a worksheet but refers to a problem contextualized in the real world. The story is that of a tourist guide, that wants to show people around, starting from and arriving at the city train station (but any other point would be fine), visiting all the roads marked in red (see Fig.2) only and only once – not to have clients seeing the same thing twice and therefore making a longer tour than needed. Is it possible to plan such a tour? Same question was posed in different situations, with different answers, as in the picture.

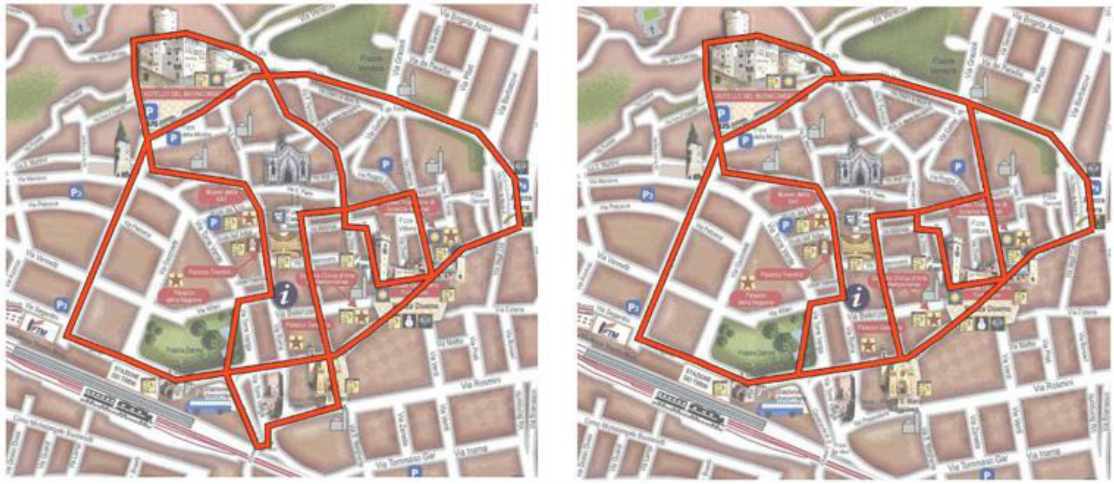

Figure 2: Task (2), with the two different situations.

Task (3) brings basically the same problem in an outdoor setting. After an introduction to orienteering maps and how to read them, the students are given a path to follow on one map of the city center (see Fig.3); the task is assigned to groups of 3 to 5 students. The goal is to follow a path which covers all the purple lines, but only once for each line, starting in the point A. “Where will the path end?” After the execution, all students which complied with the instructions will meet in point D.

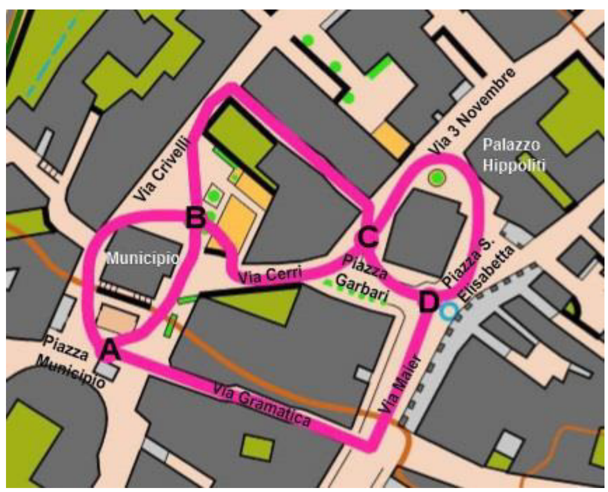

Figure 3: an example of Task (3), on a city orienteering map, with some streets highlighted.

Task (4) was a reflection, back all together, about the differences of paths taken by the various groups and what they had in common, like the end point D. “Why are we (educators) so sure that the ending point must be there?”

Task (5) is now a series of classic paper and pencil problems (as in Fig.3) about drawing a figure without taking the pen off the paper, and without re-tracing the same line more than once. Students are asked to find out which ones are possible to do, and which ones are not. Later, a discussion with the class to try to find a general rule is done, with the goal of observing some regularity about even and odd degree of the vertices of the graphs.

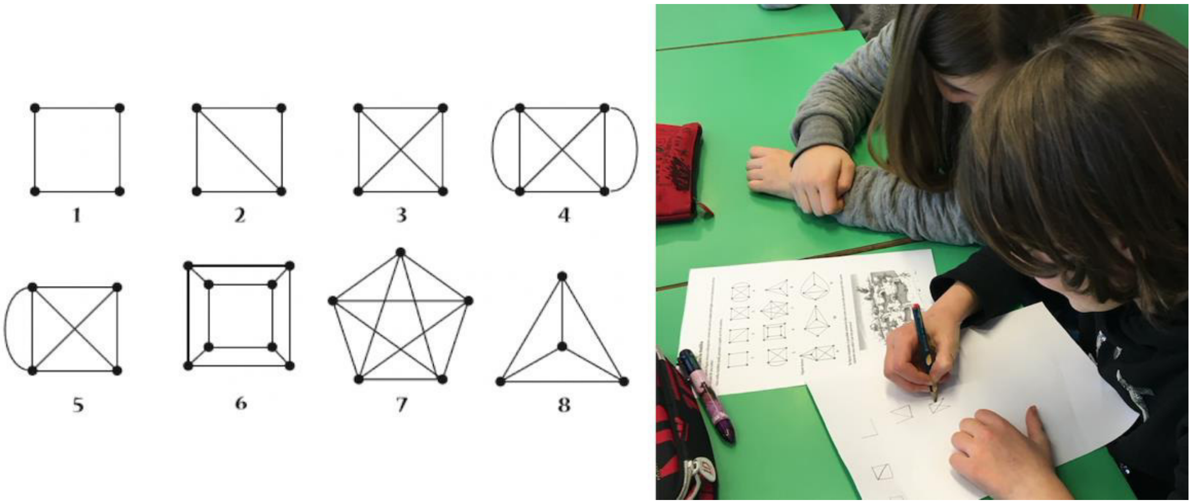

Figure 4: students involved in the paper and pencil activity, Task (4).

After agreeing on a rule that is working for each graph, in Task (6) we get back to the Königsberg bridges problem with the task of modelling it as a graph. Students, in groups, are let free to explore this task, which involves a good abstraction capability and proved not to be as easy as thought. The graph obtained allows students to prove that the Königsberg bridges problem has no solution, having all the vertices an odd degree.

Evidences from the experiments

Some observations emerged as evidence from the teaching experiments conducted. The relation between the different tasks came clear into the student's mind while performing them, even without the teachers telling them so. While working on task (5) for example we got some reactions as Lucia's one: "Wait, this is the same we were doing outside, it is like when we went around the city", referring to task (3), or Michele, during a discussion in drawing conclusions and a rule from task (5): "So this is why you were sure we would have met at the same end-point [point D in Fig. 2], there cannot be any other way!".

On some occurrences, the outdoor and RME activity worked in fostering some ideas related to the real world and movement and grounded in the use of the sensory-motor system. We observed that the outdoor activity made the students reflect in a different way than usual.

Task (3) proved to be an alternative to doing the same task on a paper. A new dimension came into play here, as showed by some of the students' reflections in task (4).

During a discussion of what happened in task (3), some students realized mathematical properties which were not obvious before (members of two different groups discussing, called 1 and 2):

Giulia (1):	We met here at point B, but we were going two different ways then.
Martin (2):	Yeah, we went from A to B, therefrom to C and then D, all the way back and then again to D. It was the easiest way.
Francesca (2):	Maybe…there was not only one way to do this, we took two different paths. Or…was your path correct? Did you follow the rules?
Giulia (1):	Yes, we did. But we found a different solution. Do you think it is correct?
Teacher:	Why didn't you just follow each other, then?

Marco (2): We could not, at that point. They [the other group] took the road from B to C, but we already used that one, so, no…

Andrea (2): Still, we all met at point D at the end, so it has to be possible that there are different solutions, but they all end up there. Why is this?

Here, the physical aspects of moving, and meeting others, leads to what will later become a conceptualization of math properties, necessary to find a general solution for the problem. In a situation elicited by the teacher discussion, for example asking why they did not follow each other until the end, when meeting in one common point, Marco answer: "we could not, at that point. They [the other group] took the road from C to D, but we already used that one, so, no…" shows that he is relating what happened in the outdoor setting with a later mathematics reflection. Finally, from a teacher interview, reflecting on the outdoor activity, the impression that outdoor learning gives a better sense of reality and generate interest and motivation in the children, together with a greater sense of responsibility, was pointed out.

DISCUSSION AND CONCLUSIONS

The changes done in the research process, placing both outdoor and practical tasks in our learning trajectory, proved to add some new challenges and to foster a better understanding of the mathematical problem. The outdoor task proved to be the best saved in the students' mind, with continuous referrals to it in the successive tasks, surely in the dimension of a positive idea of mathematics but also giving us a hint that this kind of mathematics is going towards a process of interiorization of the activities done, as RME suggests. Moreover, graph theory is a source of nice topics for classroom proposals, but sometimes requires a good deal of abstraction activity in the students.

To fill this gap, as can be seen from the evidences illustrated, principles of RME, together with body and sensory-motor experience and interaction with others, allowing students to perform what Lakoff & Núñez call metaphorical thought, through their spatial experience and concrete results obtained. Students are bringing into the mathematical setting observation they have from the real-world experience, beginning a path to "comprehend the abstract in terms of the concrete", called *conceptual metaphor*, and in this way relating their experience to the abstract mathematical problem of the graph theory rule.

This process of reconstructing the mathematical rule in a practical approach, moving in an outdoor setting and interacting with the environment and with peers in the same situation, seems to be more meaningful and the sense-making process more convincing to children, and surely worth some further research in the area.

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